

L -FUNCTIONS AND HIGHER ORDER MODULAR FORMS

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ABSTRACT. It is believed that Dirichlet series with a functional equation and Euler product of a particular form are associated to holomorphic newforms on a Hecke congruence group. We perform computer algebra experiments which find that in certain cases one can associate a kind of “higher order modular form” to such Dirichlet series. This suggests a possible approach to a proof of the conjecture.

1. INTRODUCTION

We investigate the relationship between degree-2 L -functions and modular forms. We find that degree-2 L -functions can be associated to functions on the upper half-plane which have similar properties to “second order modular forms.” Since it is conjectured that degree-2 L -functions can be associated to modular forms, this looks like a step in the right direction.

We review some classical results on modular forms and then describe the conjecture which motivates our work. A good reference for this material is Iwaniec’s book [8].

Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are integers, } ad - bc = 1, \text{ and } c \equiv 0 \pmod{N} \right\}$$

be the Hecke congruence group of level N , and suppose χ is a character mod N . The group $\Gamma_0(N)$ acts on functions $f : \mathcal{H} \rightarrow \mathbb{C}$ by $f \rightarrow f|_\gamma$ where

$$(1.1) \quad f(z) \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)^{-1} (cz + d)^{-k} f \left(\frac{az + b}{cz + d} \right).$$

Here $\mathcal{H} = \{x + iy \in \mathbb{C} : y > 0\}$ is the upper half of the complex plane. The vector space of *cusp forms of weight k and character χ for $\Gamma_0(N)$* , denoted $S_k(\Gamma_0(N), \chi)$, is the set of holomorphic functions $f : \mathcal{H} \rightarrow \mathbb{C}$ which satisfy $f|_\gamma = f$ for all $\gamma \in \Gamma_0(N)$ and which vanish at all cusps of $\Gamma_0(N)$. Since

$$(1.2) \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$$

we have $f(z) = f(z + 1)$, so there is a Fourier expansion of the form

$$(1.3) \quad f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

In the case χ is the trivial character χ_0 , the newforms in $S_k(\Gamma_0(N), \chi_0)$ have a distinguished basis of *Hecke eigenforms* which satisfy

$$(1.4) \quad f|H_N = \pm f$$

A portion of this work arose from an REU program at Bucknell University and the American Institute of Mathematics. Research supported by the American Institute of Mathematics and the National Science Foundation.

and

$$(1.5) \quad f|T_p = a_p f$$

for prime p . Here

$$H_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}$$

is the Fricke involution. If ℓ is prime,

$$(1.6) \quad T_\ell = \chi(\ell) \begin{pmatrix} \ell & \\ & 1 \end{pmatrix} + \sum_{a=0}^{\ell-1} \begin{pmatrix} 1 & a \\ & \ell \end{pmatrix},$$

is the Hecke operator. If $\ell|N$ then $\chi(\ell) = 0$ and T_ℓ is known as the Atkin-Lehner operator U_ℓ .

We will now state our motivating conjecture, and then explain its relevance to the theory of L -functions.

Conjecture 1.1. *If $f : \mathcal{H} \rightarrow \mathbb{C}$ is analytic, is periodic with period 1 (1.3), and satisfies the Fricke (1.4) and Hecke (1.5) relations with $\chi = \chi_0$, then $f \in S_k(\Gamma_0(N), \chi_0)$.*

Thus, the invariance property $f|\gamma = f$, which leads to the Fricke and Hecke relations, would actually follow from them.

We will rephrase the conjecture in terms of L -functions. Associated to a cusp form with Fourier expansion (1.3) is an L -function

$$(1.7) \quad L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Using the Mellin transform and its inverse, it can be shown that the Fricke relation (1.4) is equivalent to the functional equation

$$(1.8) \quad \xi_f(s) := \left(\frac{2\pi}{\sqrt{N}} \right)^{-s} \Gamma(s) L_f(s) = \pm (-1)^{k/2} \xi_f(k-s).$$

Also, the Hecke relations (1.5) are equivalent to $L(s, f)$ having an Euler product of the form

$$(1.9) \quad L(s, f) = \prod_p (1 - a_p p^{-s} + \chi(p) p^{k-1-2s})^{-1},$$

because both statements are equivalent to $a_{p^n m} = a_{p^n} a_m$ for $p \nmid m$ and

$$(1.10) \quad a_{p^{n+1}} = a_p a_{p^n} - \chi(p) p^{k-1} a_{p^{n-1}}.$$

Thus, Conjecture 1.1 is equivalent to

Conjecture 1.2. *If a Dirichlet series continues to an entire function of order one which is bounded in vertical strips and satisfies the functional equation (1.8) and the Euler product (1.9) with $\chi = \chi_0$, then the Dirichlet series equals $L(s, f)$ for some $f \in S_k(\Gamma_0(N), \chi_0)$.*

This conjecture should be viewed as part of the Langlands' program. Note that one does not require functional equations for twists of the L -function, as in Weil's converse theorem. As a special case, the L -function of a rational elliptic curve automatically has an Euler product of form (1.9) with $k = 2$ and $\chi = \chi_0$, so the modularity of a rational elliptic curve would follow from the analytic continuation and functional equation for one L -function.

Progress on the conjecture has been made only for small N , for the trivial character [2], and (appropriately modified) for almost the same cases for nontrivial character [7]. For $N \leq 4$, Hecke's original converse theorem establishes the conjecture. This follows from the fact that the group generated by T and H_N contains $\Gamma_0(N)$ exactly when $N \leq 4$. Note that this only uses the functional equation, not the Euler product. For larger N , one must use the Euler product in a nontrivial way. This possibility was introduced in [2], and examples were given for certain $N \leq 23$.

In this paper we specialize to the case $N = 13$, for the simple reason that this is the first case which has not been solved. Our hope is to discover some structure which can be used to attack the general case. It turns out that the $N = 13$ case leads to relations reminiscent of "higher order modular forms," which are described in the next section. In Section 3 we describe prior work and then in Section 4 we apply those methods to the case $N = 13$.

In recent work, Conrey, Odgers, Snaith, and the first author [3] have used some of the relations in this paper along with a new generalization of Weil's lemma to complete the proof for $N = 13$.

2. HIGHER ORDER MODULAR FORMS

Our discussion here is imprecise and will only convey the general flavor of this new subject. For details see [1, 4].

We first introduce some slightly simpler notation. If $f|\gamma = f$ then we have

$$(2.1) \quad \gamma \equiv 1 \pmod{\Omega_f}$$

where Ω_f is the right ideal in the group ring $\mathbb{C}[GL(2, \mathbb{R})]$ which annihilates f , the action of matrices on f being extended linearly. We will write $\gamma \equiv 1$ instead of $\gamma \equiv 1 \pmod{\Omega_f}$ throughout this paper. Thus, if f is a cusp form for the group Γ , then the invariance properties of f can be written as $f|(1 - \gamma) = 0$ for all $\gamma \in \Gamma$, or equivalently, $1 - \gamma \equiv 0$. This notation will make it easier to describe the properties of higher order modular forms.

If f is a *second order* cusp form for the group Γ , then f satisfies the relation

$$(2.2) \quad (1 - \gamma_1)(1 - \gamma_2) \equiv 0$$

for all $\gamma_1, \gamma_2 \in \Gamma$. Similarly, third order modular forms satisfy

$$(2.3) \quad (1 - \gamma_1)(1 - \gamma_2)(1 - \gamma_3) \equiv 0,$$

and so on. Roughly speaking, if f is an n th order modular form then $f|(1 - \gamma)$ is an $(n - 1)$ st order modular form. There are additional conditions involving the cusps and the parabolic elements of Γ , but our goal here is just to introduce the general idea. Indeed, it is nontrivial to determine the proper technical conditions, see [1, 4].

In connection with our exploration of Conjecture 1.1, a condition of form (2.2) will arise where γ_1 and γ_2 come from *different* groups. This first appeared in the original work of Weil on the converse theorem involving functional equations for twists. Specifically, the relation (2.2) arose where γ_2 was *elliptic of infinite order*. The following lemma applies:

Lemma 2.1. *Suppose f is holomorphic in \mathcal{H} and $\varepsilon \in GL_2(\mathbb{R})^+$ is elliptic. If $f|_k \varepsilon = f$, then either ε has finite order, or f is constant.*

This is known as "Weil's lemma" [9]. See also the discussion in Section 7.4 of Iwaniec's book [8]. By the lemma, if γ_2 is elliptic of infinite order then (2.2) implies that actually $1 - \gamma_1 \equiv 0$, which is the conclusion Weil sought.

Denote by $S_k(\Gamma_1, \Gamma_2)$ the set of analytic functions (with appropriate technical conditions) satisfying (2.2) for all $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. The above lemma says that if Γ_2 contains an elliptic element of infinite order then $S_k(\Gamma_1, \Gamma_2) = S_k(\Gamma_1)$. Note that the analyticity of f is necessary, and an analogue of Weil's converse theorem for Maass form L -functions has not been proven in classical language.

In Section 4 we will see that our assumptions on the Fricke involution and the Hecke operators lead to condition (2.2) with $\gamma_1 \in \Gamma_0(13)$ and γ_2 in some other discrete group. We also obtain higher order conditions (2.3) where each γ_j comes from a different group. This suggests the following question:

Question 2.2. *What conditions on Γ_1 and Γ_2 ensure that $S_k(\Gamma_1, \Gamma_2)$ is finite dimensional? What conditions imply that $S_k(\Gamma_1, \Gamma_2) = S_k(\Gamma_1)$?*

Part of the problem is determining the appropriate technical conditions to incorporate into the definition of $S_k(\Gamma_1, \Gamma_2)$. Even when $\Gamma_1 = \Gamma_2$ this is nontrivial. See [1, 4].

3. MANIPULATING THE HECKE OPERATORS

In [2] results were obtained for various N up to $N = 23$. The idea is to manipulate the relations $T \equiv 1$, $H_N \equiv \pm 1$ and $T_n \equiv a_n$ to obtain $\gamma \equiv 1$ for all γ in a generating set for $\Gamma_0(N)$. We will describe the cases of $N = 5, 7, 9, 11$ from [2], and then the remainder of the paper will concern the interesting relationships that arose in our exploration of the case $N = 13$.

We have the following generating sets:

$$\begin{aligned}
 \Gamma_0(N) &= \left\langle T, W_N, \begin{pmatrix} 2 & -1 \\ -N & \frac{N+1}{2} \end{pmatrix} \right\rangle, & N = 5, 7, 9, \\
 \Gamma_0(11) &= \left\langle T, W_{11}, \begin{pmatrix} 2 & -1 \\ -11 & 6 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \right\rangle, \\
 \Gamma_0(13) &= \left\langle T, W_{13}, \begin{pmatrix} 2 & -1 \\ -13 & 7 \end{pmatrix}, \begin{pmatrix} -3 & -1 \\ 13 & 4 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix} \right\rangle,
 \end{aligned}
 \tag{3.1}$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad W_N = \begin{pmatrix} 1 & \\ N & 1 \end{pmatrix}.
 \tag{3.2}$$

The generator T is for free because we have assumed a Fourier expansion. The generator W_N now follows from the Fricke relation, because $W_N = H_N T H_N$. So for these groups we have two of the generators. Note that this uses the functional equation, but not the Euler product.

In the next section we repeat the calculations from [2] in the cases $N = 5, 7, 9, 11$, and in the following sections we treat the case $N = 13$.

3.1. Levels 5, 7, 9, and 11. For every N we obtain a new generator from T_2 . This will resolve the cases $N = 5, 7$, and 9 .

Lemma 3.1 (Lemma 2 of [2]). *If $H_N \equiv \pm 1$ and $T_2 \equiv a_2$ then*

$$\begin{pmatrix} 2 & -1 \\ -N & \frac{N+1}{2} \end{pmatrix} \equiv 1.$$

Proof. Note that

$$H_N^{-1}T_2H_N = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} + \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 2 & \\ -N & 1 \end{pmatrix}.$$

Since $H_N^{-1}T_2H_N \equiv a_2H_N^{-1}H_N \equiv a_2 \equiv T_2$, we have:

$$\begin{pmatrix} 1 & \\ & 2 \end{pmatrix} + \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 2 & \\ -N & 1 \end{pmatrix} \equiv \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix}.$$

Canceling common terms from both sides we are left with

$$\begin{pmatrix} 2 & \\ -N & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix}.$$

Right multiplying by $\begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix}^{-1}$ we have

$$M_2 := \begin{pmatrix} 2 & -1 \\ -N & \frac{N+1}{2} \end{pmatrix} \equiv 1.$$

□

The lemma provides the final generator for $\Gamma_0(5)$, $\Gamma_0(7)$, and $\Gamma_0(9)$.

To obtain the final generator for $\Gamma_0(11)$ we will combine the Hecke operators T_3 and T_4 . For T_3 we have

$$\begin{aligned} 0 &\equiv H_N(T_3 - a_3)H_N - (T_3 - a_3) \\ &= -\begin{pmatrix} 1 & 1 \\ & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ & 3 \end{pmatrix} + \begin{pmatrix} 3 & \\ -N & 1 \end{pmatrix} + \begin{pmatrix} 3 & \\ -2N & 1 \end{pmatrix} \\ (3.3) \quad &\equiv -\begin{pmatrix} 1 & 1 \\ & 3 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ & 3 \end{pmatrix} + \begin{pmatrix} 3 & \\ -N & 1 \end{pmatrix} + \begin{pmatrix} 3 & \\ N & 1 \end{pmatrix}, \end{aligned}$$

where the second step used

$$(3.4) \quad \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} \equiv 1 \quad \text{and} \quad \begin{pmatrix} 1 & \\ N & 1 \end{pmatrix} \equiv 1.$$

We can combine the terms in pairs using

$$\begin{pmatrix} 1 & a \\ & p \end{pmatrix} - \begin{pmatrix} p & \\ Nb & 1 \end{pmatrix} = \left(1 - \begin{pmatrix} p & -a \\ Nb & \frac{-Nab+1}{p} \end{pmatrix}\right) \begin{pmatrix} 1 & a \\ & p \end{pmatrix}$$

to get

$$(3.5) \quad \left(1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}\right) \beta(1/3) + \left(1 - \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix}\right) \beta(-1/3) \equiv 0,$$

where $\beta(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$. We will combine this with a relation obtained from T_4 .

Since T_4 and T_2 are not independent, there is more than one way to proceed. The calculation which seems most natural to us begins with

$$\begin{aligned} 0 &\equiv H_N(T_4 - a_4)H_N - (T_4 - a_4) \\ &\quad - [H_N(T_2 - a_2)H_N - (T_2 - a_2)] \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} \\ &\quad - [H_N(T_2 - a_2)H_N - (T_2 - a_2)] \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \end{aligned}$$

$$(3.6) \quad = - \begin{pmatrix} 1 & 1 \\ & 4 \end{pmatrix} + \begin{pmatrix} 4 & \\ -3N & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ & 4 \end{pmatrix} + \begin{pmatrix} 4 & \\ -N & 1 \end{pmatrix}.$$

Combining terms as in the T_3 case gives

$$(3.7) \quad \left(1 - \begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix}\right) \beta(1/4) + \left(1 - \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}\right) \beta(-1/4) \equiv 0.$$

Combining (3.5) and (3.7) we obtain

$$(3.8) \quad \begin{aligned} 1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} &\equiv - \left(1 - \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix}\right) \beta\left(-\frac{2}{3}\right) \\ &= \left(1 - \begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix}\right) \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \beta\left(-\frac{2}{3}\right) \\ &\equiv - \left(1 - \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}\right) \beta\left(-\frac{2}{4}\right) \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \beta\left(-\frac{2}{3}\right) \\ &= \left(1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}\right) \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} \beta\left(-\frac{2}{4}\right) \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \beta\left(-\frac{2}{3}\right). \end{aligned}$$

However,

$$\begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} \beta\left(-\frac{2}{4}\right) \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \beta\left(-\frac{2}{3}\right) = \begin{pmatrix} 1 & -2/3 \\ 11/2 & -8/3 \end{pmatrix}$$

is elliptic but not of finite order. So by Lemma 2.1,

$$\begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \equiv 1.$$

This is the final generator for $\Gamma_0(11)$.

4. LEVEL 13, MIMIC PREVIOUS METHODS

We will mimic the method used for $\Gamma_0(11)$ for $\Gamma_0(13)$, but things will not work out as nicely. What will arise is an expression of the form (2.2) that appears in the definition of second order modular form.

4.1. The case of T_3 . From T_3 we obtain the following expression, which is analogous to (3.5),

$$(4.1) \quad \left(1 - \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix}\right) \beta(1/3) + \left(1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right) \beta(-1/3) \equiv 0.$$

We manipulate this similarly to the example for $\Gamma_0(11)$:

$$(4.2) \quad \begin{aligned} 1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} &\equiv - \left(1 - \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix}\right) \beta\left(\frac{2}{3}\right) \\ &= \left(1 - \begin{pmatrix} -4 & 1 \\ -13 & 3 \end{pmatrix}\right) \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix} \beta\left(\frac{2}{3}\right) \\ &\equiv \left(1 - H_{13} \begin{pmatrix} -4 & 1 \\ -13 & 3 \end{pmatrix} H_{13}\right) H_{13} \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix} \beta\left(\frac{2}{3}\right) \\ &= \left(1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right) H_{13} \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix} \beta\left(\frac{2}{3}\right). \end{aligned}$$

So,

$$(4.3) \quad \left(1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right) (1 - \varepsilon_1) \equiv 0$$

where

$$(4.4) \quad \varepsilon_1 = H_{13} \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix} \beta \left(\frac{2}{3}\right) = \begin{pmatrix} \sqrt{13} & \frac{14}{3\sqrt{13}} \\ -3\sqrt{13} & -\sqrt{13} \end{pmatrix}.$$

Since ε_1 is elliptic of order 2 we cannot obtain anything from Lemma 2.1. However, we do have an expression of the form (2.2) which looks like the definition of a second order modular form.

4.2. The case of T_4 . From T_4 , again proceeding as in the $\Gamma_0(11)$ example, we first have

$$(4.5) \quad \left(1 - \begin{pmatrix} 4 & -1 \\ 13 & -3 \end{pmatrix}\right) \beta(1/4) + \left(1 - \begin{pmatrix} 4 & 1 \\ -13 & -3 \end{pmatrix}\right) \beta(-1/4) \equiv 0.$$

Continuing exactly as above, this leads to

$$(4.6) \quad \left(1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right) (1 - \varepsilon_2) \equiv 0$$

where

$$(4.7) \quad \varepsilon_2 = \begin{pmatrix} -\sqrt{13} & \frac{-4}{\sqrt{13}} \\ \frac{7\sqrt{13}}{2} & \sqrt{13} \end{pmatrix}.$$

Again ε_2 is elliptic of order 2.

4.3. Combining T_3 and T_4 . We can combine the two relationships to obtain

$$(4.8) \quad 0 \equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right] (1 - \varepsilon)$$

for any ε in the group generated by ε_1 and ε_2 , and perhaps one of those elements will be elliptic of infinite order? Unfortunately, this is not the case. Note that

$$\varepsilon_1 \varepsilon_2 = \begin{pmatrix} \frac{10}{3} & \frac{2}{3} \\ -\frac{13}{2} & -1 \end{pmatrix},$$

which is hyperbolic. Since ε_1 and ε_2 have order 2, the group they generate contains only the elements $(\varepsilon_1 \varepsilon_2)^n$ and $\varepsilon_2 (\varepsilon_1 \varepsilon_2)^n$, so that group is discrete.

Although T_3 and T_4 were not sufficient to obtain the missing generator, there are an infinite number of other Hecke operators to try.

4.4. The case of T_6 . We now proceed with similar calculations with T_6 . We have

$$(4.9) \quad \begin{aligned} 0 &\equiv H_{13}(T_6 - a_6)H_{13} - (T_6 - a_6) \\ &\quad - [H_{13}(T_2 - a_2)H_{13} - (T_2 - a_2)] \begin{pmatrix} 3 & \\ & 1 \end{pmatrix} - [H_{13}(T_2 - a_2)H_{13} - (T_2 - a_2)] \begin{pmatrix} 1 & \\ & 3 \end{pmatrix} \\ &\quad - [H_{13}(T_3 - a_3)H_{13} - (T_3 - a_3)] \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} - [H_{13}(T_3 - a_3)H_{13} - (T_3 - a_3)] \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \\ &\equiv - \begin{pmatrix} 1 & 1 \\ 6 & \end{pmatrix} + \begin{pmatrix} 6 & \\ -65 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ 6 & \end{pmatrix} + \begin{pmatrix} 6 & \\ -13 & 1 \end{pmatrix}. \end{aligned}$$

Using manipulations similar to those above gives

$$\begin{aligned} 0 &\equiv \left[-1 + \begin{pmatrix} 6 & -1 \\ 65 & 11 \end{pmatrix} \right] \begin{pmatrix} 1 & 1 \\ 6 & 6 \end{pmatrix} + \left[-1 + \begin{pmatrix} 6 & -5 \\ -13 & 11 \end{pmatrix} \right] \begin{pmatrix} 1 & 5 \\ 6 & 6 \end{pmatrix} \\ &\equiv \left[-1 + \begin{pmatrix} 6 & -1 \\ 65 & 11 \end{pmatrix} \right] \begin{pmatrix} 1 & 1 \\ 6 & 6 \end{pmatrix}, \end{aligned}$$

because $\begin{pmatrix} 6 & -5 \\ -13 & 11 \end{pmatrix} = M_2^{-1}H_{13}T^{-1}H_{13}T^{-1}$ so the second term on the first line is $\equiv 0$. So we have

$$\begin{pmatrix} 1 & 1 \\ 6 & 6 \end{pmatrix} + \begin{pmatrix} 6 & -1 \\ -65 & 11 \end{pmatrix} \equiv 0,$$

so

$$\begin{pmatrix} 6 & -1 \\ -65 & 11 \end{pmatrix} \equiv 1$$

This is not a new matrix because $\begin{pmatrix} 6 & -1 \\ -65 & 11 \end{pmatrix} = H_{13}TH_{13}TH_{13}M_2H_{13}$. That is, the above manipulations with T_6 produce results that can be obtained from T_2 .

4.5. Computer manipulation of Hecke operators. The explicit manipulation of Hecke operators described in this paper are quite tedious to do by hand, so we decided to make use of a computer. We modified Mathematica to do calculations in the group ring $\mathbb{C}[SL(2, \mathbb{R})]$, made functions for the Hecke operators, automated manipulations that occur repeatedly (such as the first step in every example in the previous section of this paper), and implemented some crude simplifications procedures.

For the simplification procedures, we sought to automate the discovery, for example, that if $T \equiv 1$, $H_{13} \equiv \pm 1$, and $M_2 \equiv 1$, then

$$(4.11) \quad -1 + \begin{pmatrix} 6 & -1 \\ 65 & 11 \end{pmatrix} \equiv 0,$$

as we saw at the end of the previous section. Our approach was to put all of the matrices in each expression in “simplest form” by considering all products (on the left) with, for example, fewer than 6 matrices where are known to be $\equiv 1$, and then keeping the representative which has the smallest entries. This idea worked surprisingly well.

We also implemented a “factorization” function which would do the (trivial) calculation to check such things as whether $1 - \gamma_1 - \gamma_2 + \gamma_3$ was of the form $(1 - \gamma_1)(1 - \gamma_2)$ or $(1 - \gamma_2)(1 - \gamma_1)$.

4.6. The case of T_7 . Calculations with T_7 yield interesting results. We have

$$\begin{aligned} 0 &\equiv H_{13}(T_7 - a_7)H_{13} - (T_7 - a_7) \\ &\equiv -\begin{pmatrix} 1 & 2 \\ 7 & 7 \end{pmatrix} + \begin{pmatrix} 7 & 3 \\ -52 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 7 & 7 \end{pmatrix} + \begin{pmatrix} 7 & 1 \\ -65 & 1 \end{pmatrix} \\ &\quad - \begin{pmatrix} 1 & 4 \\ 7 & 7 \end{pmatrix} + \begin{pmatrix} 7 & 1 \\ -26 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ 7 & 7 \end{pmatrix} + \begin{pmatrix} 7 & 1 \\ -39 & 1 \end{pmatrix}. \end{aligned}$$

Note that the expression on the right consists of 4 pair of matrices, as opposed to the 6 pair that one would expect to obtain from T_7 . This is because two pair canceled during simplification.

It turns out that the right side of the above expression factors as

$$\begin{aligned}
& \left[-1 + \begin{pmatrix} -3 & 1 \\ -13 & 4 \end{pmatrix} \right] \begin{pmatrix} 1 & 2 \\ 7 & 7 \end{pmatrix} + \left[-1 + \begin{pmatrix} 7 & 4 \\ -65 & -37 \end{pmatrix} \right] \begin{pmatrix} 1 & -4 \\ 7 & 7 \end{pmatrix} \\
& + \left[-1 + \begin{pmatrix} 7 & -4 \\ -26 & 15 \end{pmatrix} \right] \begin{pmatrix} 1 & 4 \\ 7 & 7 \end{pmatrix} + \left[-1 + \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] \begin{pmatrix} 1 & -2 \\ 7 & 7 \end{pmatrix} \\
& = \left[-\begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} + 1 \right] \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}^{-1} H_{13} \begin{pmatrix} 1 & 2 \\ 7 & 7 \end{pmatrix} \\
& + \left[-\begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} + 1 \right] \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}^{-1} H_{13} \begin{pmatrix} 7 & -1 \\ -13 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -4 \\ 7 & 7 \end{pmatrix} \\
& + \left[-1 + \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] \begin{pmatrix} 7 & 1 \\ 13 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 4 \\ 7 & 7 \end{pmatrix} + \left[-1 + \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] \begin{pmatrix} 1 & -2 \\ 7 & 7 \end{pmatrix} \\
& = \left[-1 + \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] \\
& \quad \left(-\begin{pmatrix} -\sqrt{13} & \frac{2}{\sqrt{13}} \\ 3\sqrt{13} & -\sqrt{13} \end{pmatrix} - \begin{pmatrix} 2\sqrt{13} & \frac{1}{\sqrt{13}} \\ -7\sqrt{13} & \frac{1}{\sqrt{13}} \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ -13 & -3 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 7 & 7 \end{pmatrix} \right).
\end{aligned}$$

We can right multiply by the inverse of any of the four matrices in the second factor to rewrite this in the form $(1 - \gamma)(1 + A - B - C)$. For no good reason we choose the first term, giving

$$\begin{aligned}
0 & \equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] \\
& \quad \times \left(1 + \begin{pmatrix} \frac{29}{7} & \frac{5}{7} \\ -13 & -2 \end{pmatrix} - \begin{pmatrix} \frac{5\sqrt{13}}{7} & \frac{17}{7\sqrt{13}} \\ \frac{-22\sqrt{13}}{7} & \frac{-5\sqrt{13}}{7} \end{pmatrix} - \begin{pmatrix} \frac{5\sqrt{13}}{7} & \frac{24}{7\sqrt{13}} \\ -3\sqrt{13} & -\sqrt{13} \end{pmatrix} \right) \\
(4.14) \quad & = \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] (1 + A - B - C),
\end{aligned}$$

say. This expression factors further. Specifically, one can check that $A = CB$, so we have

$$(4.15) \quad 0 \equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] (1 - C)(1 - B)$$

Unfortunately, $B^2 = 1$, so we cannot immediately cancel the final factor to reduce to a second-order type expression. It would be good if that happened, because we would have another matrix to combine with the ε_1 and ε_2 from Sections 4.1 and 4.2.

However, there is a curious benefit to having $B^2 = 1$, for we also have $AB = C$, so

$$(4.16) \quad 0 \equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] (1 - A)(1 - B).$$

Note that if $B^2 = 1$, independent of any conditions on A and C , then $(1 + A - B - C)(1 + B) = (1 - CA^{-1})(1 + ABA^{-1})A$, so

$$(4.17) \quad 0 \equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] (1 - CA^{-1})(1 + ABA^{-1}),$$

which is almost a third-order condition. Such expressions arise whenever we have an order-2 matrix, so some types of factorization are not a surprise. In the particular case at hand,

$CA^{-1} = ABA^{-1}$, which has order 2, so $(1 - CA^{-1})(1 + ABA^{-1}) = 0$ and (4.17) contains absolutely no information. Perhaps one should think that if $B^2 = 1$ then there always is some factorization, for either (4.17) is nontrivial, or the expression factors nontrivially in another way.

4.7. A few other cases. From T_{10} we get

$$\begin{aligned}
 0 &\equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] \\
 &\quad \times \left(1 + \begin{pmatrix} \frac{21}{5} & \frac{2}{5} \\ -13 & -1 \end{pmatrix} - \begin{pmatrix} \frac{2\sqrt{13}}{5} & \frac{7}{5\sqrt{13}} \\ \frac{-11\sqrt{13}}{5} & \frac{-2\sqrt{13}}{5} \end{pmatrix} - \begin{pmatrix} \frac{4\sqrt{13}}{5} & \frac{19}{5\sqrt{13}} \\ -3\sqrt{13} & -\sqrt{13} \end{pmatrix} \right) \\
 (4.18) \quad &= \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] (1 + A - B - C),
 \end{aligned}$$

say. Again $A = CB$ and $B^2 = 1$, so we obtain two factorizations.

From T_{15} we get

$$\begin{aligned}
 0 &\equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] \\
 &\quad \times \left(1 + \begin{pmatrix} \frac{16}{5} & 1 \\ -\frac{117}{5} & -7 \end{pmatrix} - \begin{pmatrix} 4\sqrt{13} & \frac{15}{\sqrt{13}} \\ \frac{-209\sqrt{13}}{15} & -4\sqrt{13} \end{pmatrix} - \begin{pmatrix} \frac{17\sqrt{13}}{15} & \frac{4}{\sqrt{13}} \\ \frac{-59\sqrt{13}}{15} & -\sqrt{13} \end{pmatrix} \right),
 \end{aligned}$$

which again factors in the same two ways.

From T_9 we get

$$\begin{aligned}
 0 &\equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} \right] \\
 &\quad \times \left(1 + \begin{pmatrix} \frac{10}{3} & 1 \\ -\frac{13}{3} & -1 \end{pmatrix} - \begin{pmatrix} 2\sqrt{13} & \frac{9}{\sqrt{13}} \\ \frac{-53\sqrt{13}}{9} & -2\sqrt{13} \end{pmatrix} - \begin{pmatrix} \frac{7\sqrt{13}}{9} & \frac{4}{\sqrt{13}} \\ \frac{-25\sqrt{13}}{9} & -\sqrt{13} \end{pmatrix} \right).
 \end{aligned}$$

which again factors in the same two ways.

It would be helpful to understand the underlying reason why these expressions factor.

More time on the computer should produce more relations, but it is not clear how they will combine to produce the desired result. It would be interesting if the relations could build to the point where one could reduce higher order relations to lower order ones, which could then combine with previously found relations to cause additional cancellation, and so on, reducing down to the one missing generator for $\Gamma_0(13)$. It would be more satisfying if one could find manipulations which produce any specific matrix, as one does in the proof of Weil's converse theorem.

Our approach here is to look for factorizations $(1 - \gamma)(1 - \delta)(1 - \varepsilon) \equiv 0$ in the hopes of eliminating the last factor, perhaps because ε is elliptic of infinite order. In the case of expressions that do not factor, it would be interesting to know if there are cancellation laws beyond those implied by Weil's lemma. That is, are there conditions on A, B, C such that $f|(1 + A - B - C) = 0$ implies some apparently stronger condition on f , beyond those cases where $1 + A - B - C$ factors and Weil's lemma applies?

4.8. A curiosity. All the manipulations in this paper involve “pairing up” the terms in a linear combination of matrices. Usually there is a natural way to do this, for one is hoping to produce matrices in $\Gamma_0(N)$. However, it is possible to pair the matrices in different ways, and one would like some justification for the choices and to know the consequences of making the right (or wrong) choices. This is discussed extensively in [6].

We now give an example by repeating the analysis of Section 3 making the wrong choices. From (3.3) with $N = 11$ we have

$$(4.21) \quad \left(1 - \begin{pmatrix} 3 & -1 \\ 11 & -\frac{10}{3} \end{pmatrix}\right) \beta(1/3) + \left(1 - \begin{pmatrix} 3 & 1 \\ -11 & -\frac{10}{3} \end{pmatrix}\right) \beta(-1/3) \equiv 0,$$

where $\beta(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$. Now doing manipulations exactly as in Section 4.1 we obtain

$$(4.22) \quad 0 \equiv \left(1 - \begin{pmatrix} 3 & -1 \\ 11 & -\frac{10}{3} \end{pmatrix}\right) (1 - \varepsilon),$$

where

$$(4.23) \quad \varepsilon = H_{11} \begin{pmatrix} 3 & 1 \\ -11 & -10/3 \end{pmatrix} \beta(-2/3) = \begin{pmatrix} \sqrt{11} & -\frac{4}{\sqrt{11}} \\ 3\sqrt{11} & -\sqrt{11} \end{pmatrix},$$

which has order 2.

Note that the above manipulations cannot lead to $\begin{pmatrix} 3 & -1 \\ 11 & -\frac{10}{3} \end{pmatrix} \equiv 1$. Indeed, if p is prime, the group generated by $\Gamma_0(p)$ and H_p is a maximal discrete subgroup of $SL(2, \mathbb{R})$. So no manipulation can lead to a new matrix which is $\equiv 1$. Yet, we do obtain additional second order modular form type properties for newforms in $S_k(\Gamma_0(11))$. It is not clear what mechanism will lead to the production of new matrices for $N = 13$, yet not produce a contradiction when $N = 11$.

Using T_4 in the same way gives

$$(4.24) \quad 0 \equiv \left(1 - \begin{pmatrix} 4 & -1 \\ 11 & -\frac{5}{2} \end{pmatrix}\right) \left(1 - \begin{pmatrix} \sqrt{11} & -\frac{3}{\sqrt{11}} \\ 4\sqrt{11} & -\sqrt{11} \end{pmatrix}\right),$$

and from T_6 you get

$$(4.25) \quad 0 \equiv \left(1 - \begin{pmatrix} 6 & -1 \\ 11 & -\frac{5}{3} \end{pmatrix}\right) \left(1 - \begin{pmatrix} \sqrt{11} & -\frac{2}{\sqrt{11}} \\ 6\sqrt{11} & -\sqrt{11} \end{pmatrix}\right),$$

where the inner matrix is hyperbolic.

This illustrates that $f|(1 - \varepsilon)(1 - \delta) = 0$ need not imply f is constant, and even having multiple independent relations of that form is not sufficient. In the case here, we have the above relations in addition to $f|(1 - \gamma)$ for all $\gamma \in \Gamma_0(11)$. This suggest that these “second order” conditions may be weaker than they appear.

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